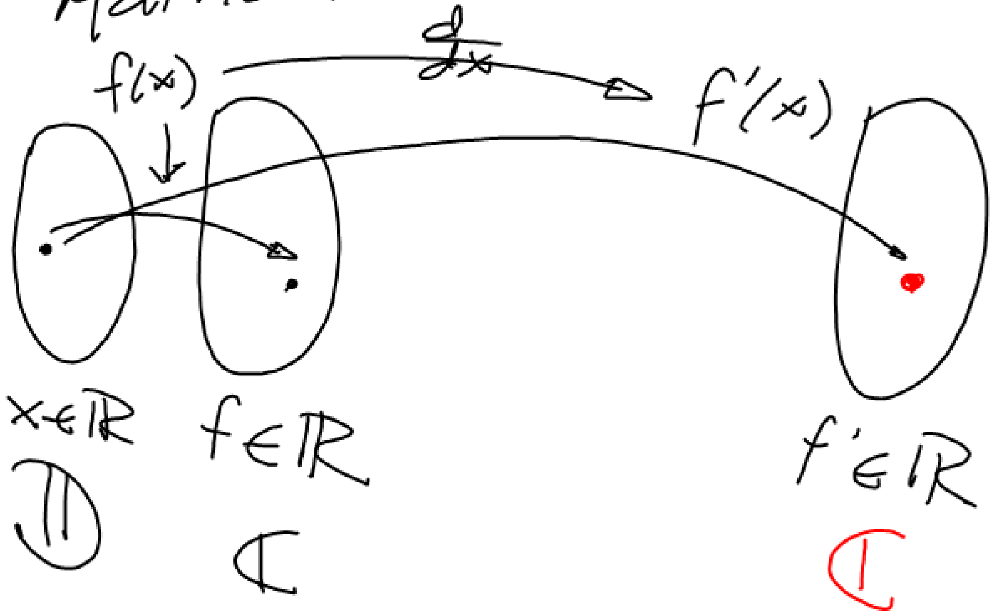
Chapter 4: Laplace Transform as method for solve Initial Condition of Linear Ordinary Differential Equation.

What is Transform in Mathematics?

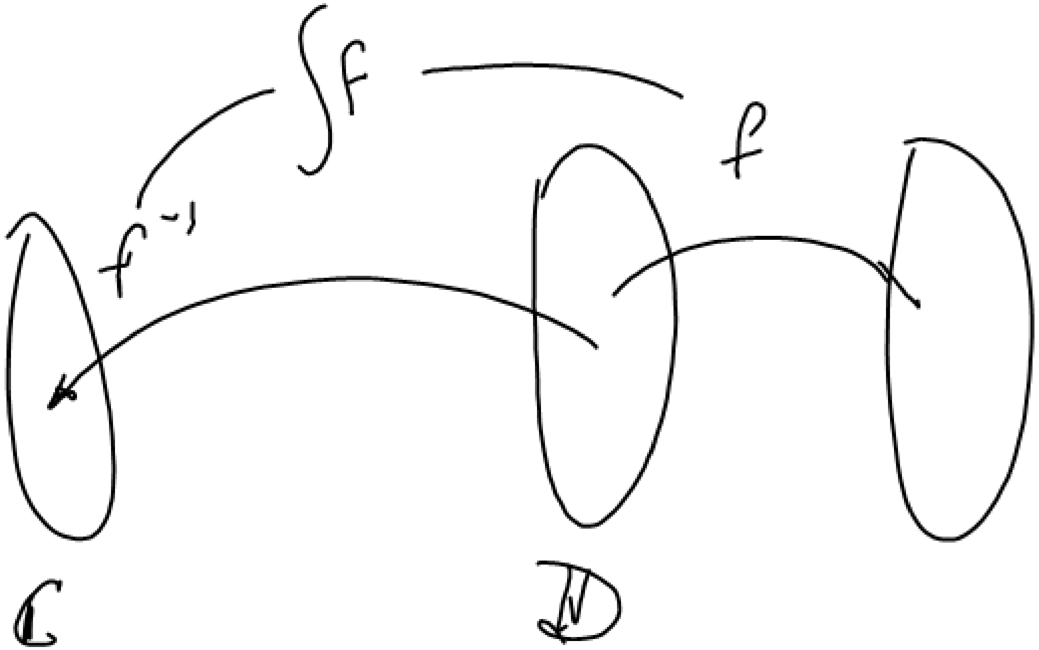


Real function of Real unknown Variable

$$f(x) = x^3 \longrightarrow f'(x) = 3x^2$$

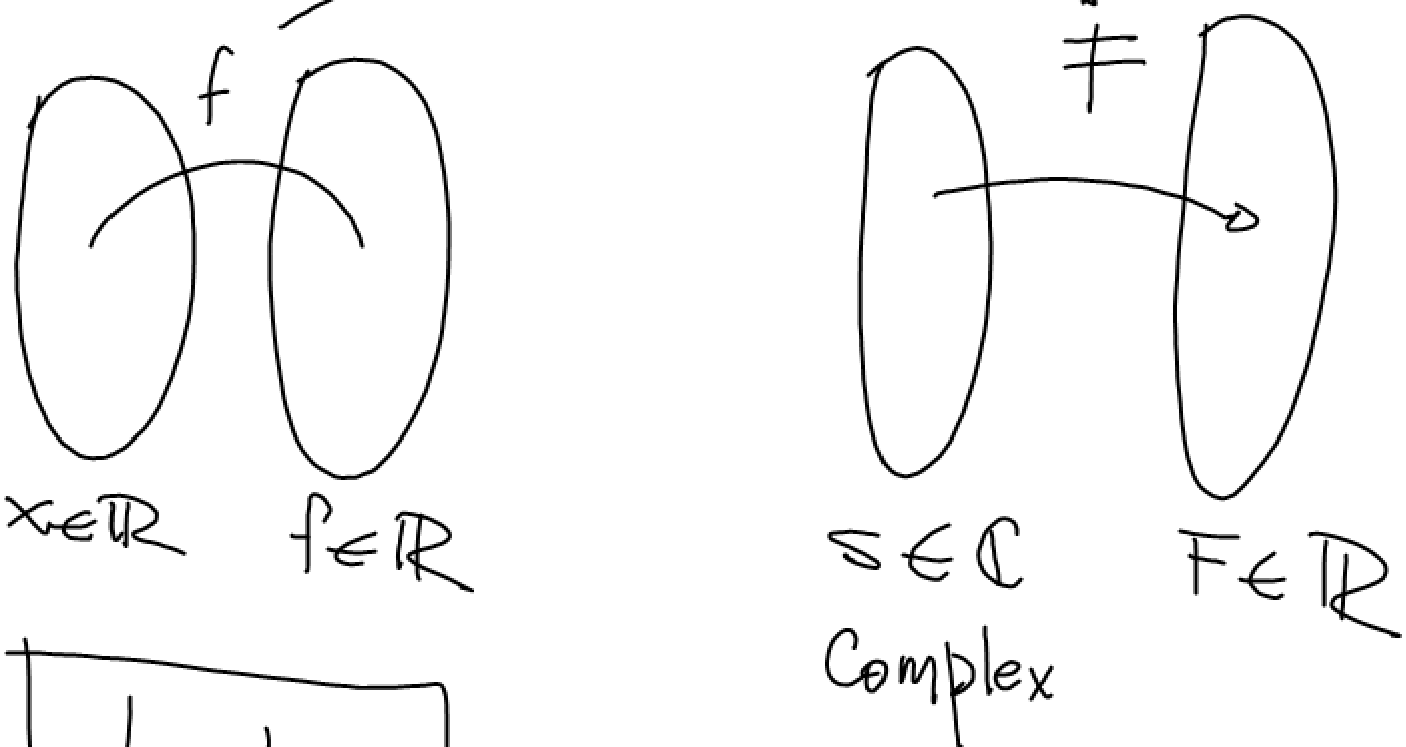
D

x	$f(x)$	$f'(x)$
2	8	12



$$\text{Operators} \begin{cases} \dfrac{d}{dx} \\ \int dx \end{cases}$$

$$T\{f\} = F$$



$\boxed{\text{Laplace}}$

with(inttrans)

$$T\{f\} = F$$

$a, b \in \mathbb{R}$

$$(af + bg) \xrightarrow[\text{Linear}]{} aF + bG$$

$$\frac{df}{dx} \longrightarrow sF$$

$$\int f\,dx \longleftarrow \frac{F}{s}$$

$$T\{f(t)\} = \int_{-\infty}^{\infty} N(t,s)\, f(t)\, dt$$

definition

<u>Laplace</u> Transform.

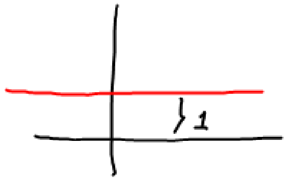
$$N(t,s) = \begin{cases} 0 \; ; & t < 0 \\ e^{-st} \; ; & t \geq 0 \end{cases}$$

gothic

$\mathcal{L}$

$$\boxed{\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)\, dt}$$

$f(t)=1$



$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\cdot(1)\cdot dt$$

$$= \left[\int e^{-st}\,dt\right]_0^\infty$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left(\lim_{b\to\infty} e^{-sb} - 1\right)$$

0

$$\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} \frac{1}{e^{sb}}$$

$$\lim_{b\to\infty} e^{-sb} = \lim_{a\to\infty} \frac{1}{a} = 0$$

zero

$$\mathcal{L}\{1\} = -\frac{1}{s}(0-1) \Rightarrow \frac{1}{s}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \left[\int t e^{-st}\, dt\right]_0^\infty$$

$\int t e^{-st}\, dt$ $\quad u = t \quad du = dt$

$\quad dv = e^{-st}\, dt \quad v = \frac{e^{-st}}{-s}$

$$\int u\, dv = u \cdot v - \int v\, du$$

$$\int t e^{-st}\, dt = \frac{t e^{-st}}{-s} - \int \frac{e^{-st}\, dt}{-s}$$

$$= -\frac{1}{s}\left(t e^{-st}\right) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)$$

$$\int t e^{-st}\, dt = -\frac{1}{s}\left(t e^{-st}\right) - \frac{1}{s^2} e^{-st}.$$

$$\mathcal{L}\{t\} = -\frac{1}{s}\left[t e^{-st}\right]_0^\infty - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{1}{s}\left[\lim_{b\to\infty} b \cdot \lim_{b\to\infty} e^{-sb} - (0)(1)\right] - \frac{1}{s^2}\left((0) - 1\right)$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{t^3\} = \frac{3!}{s^4} \Rightarrow \frac{6}{s^4}$$

$$f = e^{at}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \left[\int e^{-(s-a)t}\, dt\right]_0^\infty$$

$$= \left[-\frac{1}{s-a}\left(e^{-(s-a)t}\right)\right]_0^\infty$$

$$= -\frac{1}{s-a}\left(e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{s-a}\left(\cancelto{0}{\lim_{t\to\infty} e^{-(s-a)t}} - (1)\right)$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[\int e^{-st} f'(t)\,dt\right]_0^\infty$$

$$\int e^{-st} f'(t)\,dt = f(t)e^{-st} - \int f(t)(-se^{-st})\,dt$$

$u=e^{-st}$ $\quad du=-se^{-st}dt$

$dv=f'(t)dt$ $\quad v=f(t)$

$\int u\,dv = uv - \int v\,du.$

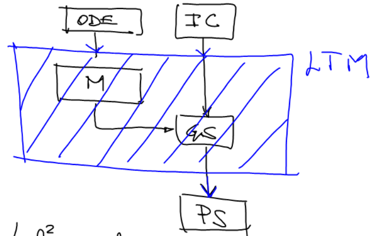
$$\mathcal{L}\{f'(t)\} = \left[f(t)e^{-st} + s\int e^{-st}f(t)\,dt\right]_0^\infty$$

$$\mathcal{L}\{f'(t)\} = \left(\lim_{t\to\infty} f(t)\, \overset{\to 0}{\lim_{t\to\infty} e^{-st}} - f(0)(1)\right) + s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0 \qquad y(0) = 2, \quad y'(0) = -2$$



$$\mathcal{L}\left\{\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y\right\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 5\mathcal{L}\left\{\frac{dy}{dt}\right\} + 6\mathcal{L}\{y\} = 0\cdot\mathcal{L}\{1\}$$

$$\left(s^2\mathcal{L}\{y\} - s(2) - (-2)\right) - 5\left(s\mathcal{L}\{y\} - (2)\right) + 6\mathcal{L}\{y\} = 0$$

$$\left(s^2 - 5s + 6\right)\mathcal{L}\{y\} - 2s + 12 = 0$$

$$\left(s^2 - 5s + 6\right)\mathcal{L}\{y\} = 2s - 12$$

$$\boxed{\mathcal{L}\{y\} = \frac{2s-12}{s^2 - 5s + 6}}$$

Laplace Transform of Particular Solution

$$\frac{2s-12}{(s-2)(s-3)} = \frac{A}{(s-2)} + \frac{B}{(s-3)}$$

$$2s - 12 = A(s-3) + B(s-2)$$

if $s = 2$

$$2(2) - 12 = A(2-3) + (0) \qquad -A = -8 \qquad \boxed{A = 8}$$

if $s = 3$

$$2(3) - 12 = (0) + B(3-2) \qquad \boxed{B = -6}$$

$$\mathcal{L}\{y\} = \frac{8}{s-2} - \frac{6}{s-3}$$

$$y = \mathcal{L}^{-1}\left\{\frac{8}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{6}{s-3}\right\}$$

$$y = 8\left(e^{2t}\right) - 6\left(e^{3t}\right)$$

$$\boxed{y = 8e^{2t} - 6e^{3t}}$$